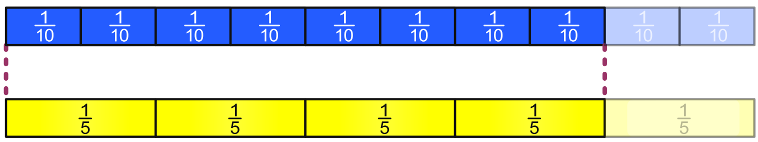
1 Simplifying Fractions

1.1 Definition

To enlarge a fraction, we multiply the numerator and the denominator of this fraction by the same natural number (except 0). This gives us a fraction of equal size. The “reverse” process is also possible.
If the numerator and the denominator of a fraction have a common factor, we can divide both by that factor. This gives us a fraction of equal size.

Let’s consider the fraction $\mathbf{\frac{8}{10}}$. As we can see, there is a fraction that is equal in size to $\mathbf{\frac{8}{10}}$, but with a smaller numerator and denominator, namely $\mathbf{\frac{4}{5}}$.



If we divide the denominator 10 by 2, we get 5. $\mathbf{\frac{1}{5}}$ is twice as large as $\mathbf{\frac{1}{10}}$.
To obtain an equal-sized fraction, we therefore only need half as many parts. So we also divide the numerator 8 by 2 and get 4. We have:

$$\frac{8}{10} = \frac{8 \div 2}{10 \div 2} = \frac{4}{5}\ .$$

We can express this connection somewhat imprecisely, but very briefly, as follows:

The bigger the parts, the fewer in number.

The process of dividing the numerator and the denominator of a fraction by a common factor of numerator and denominator is called **simplifying**. Simplifying a fraction results in a fraction of equal size.

> If the numerator and the denominator of a fraction are divided by a common factor, a fraction of equal size results. This process is called **simplifying**.

We can also express this connection very briefly:

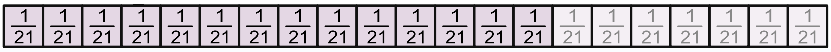
$$\boxed{\frac{a}{b} = \frac{a \div n}{b \div n}}$$

Here, $\mathbf{\frac{a}{b}}$ stands for any fraction and **n** stands for any common factor of **a** and **b**. To avoid nonsense, we will from now on divide only by such common factors **n** that are greater than **1**.

1.2 Examples

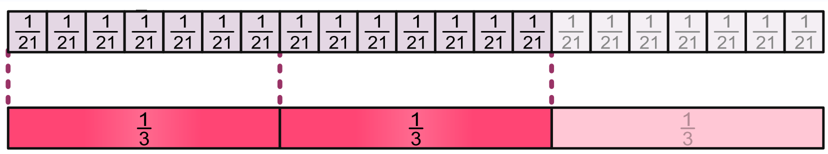
Example 1

We want to simplify the fraction $\mathbf{\frac{14}{21}}$ by 7.



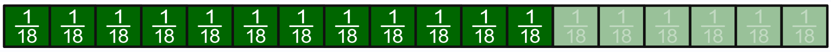
If we divide the denominator 21 by 7, we get 3.
$\mathbf{\frac{1}{3}}$ is seven times as large as $\mathbf{\frac{1}{21}}$.
To get a fraction of equal size,
we therefore only need one seventh as many parts.
If we divide the numerator 14 by 7, we get 2.

$$\frac{14}{21} = \frac{14 \div 7}{21 \div 7} = \frac{2}{3}$$



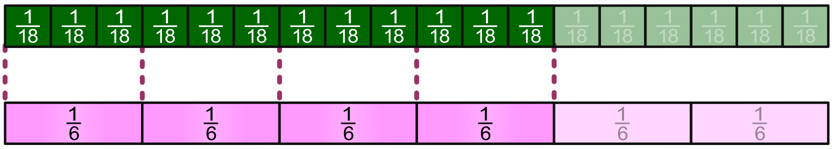
Example 2

We want to simplify the fraction $\frac{\mathbf{12}}{\mathbf{18}}$ by 3.



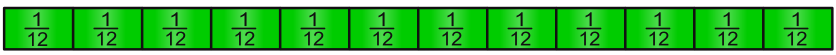
If we divide the denominator 18 by 3, we get 6.
$\frac{\mathbf{1}}{\mathbf{6}}$ is three times as large as $\frac{\mathbf{1}}{\mathbf{18}}$.
To get a fraction of equal size, we therefore only need one third as many parts.
If we divide the numerator 12 by 3, we get 4.

$$\frac{12}{18} = \frac{12 \div 3}{18 \div 3} = \frac{4}{6}\ .$$

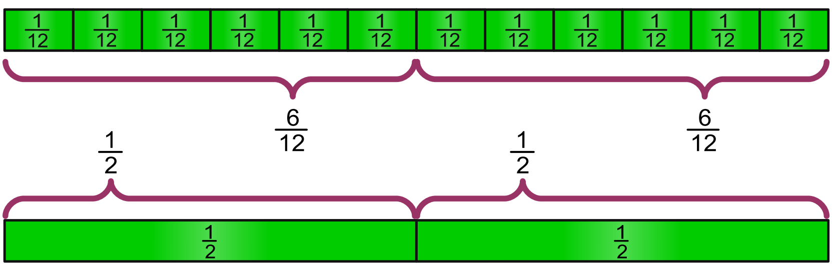


1.3 Simplifying by all divisors of the denominator

For every divisor of a denominator, there are fractions that we can simplify by this divisor. Let's look at an example: Here we have 12 twelfths.

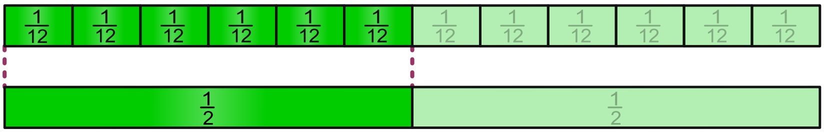


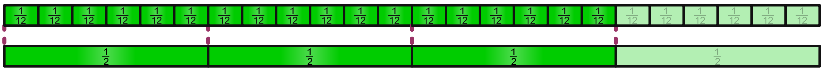
Since 12 is divisible by 6, we can group 6 twelfths into *one* part each. On this fraction strip, there are $12 \div 6 = 2$ such parts. So these are halves.



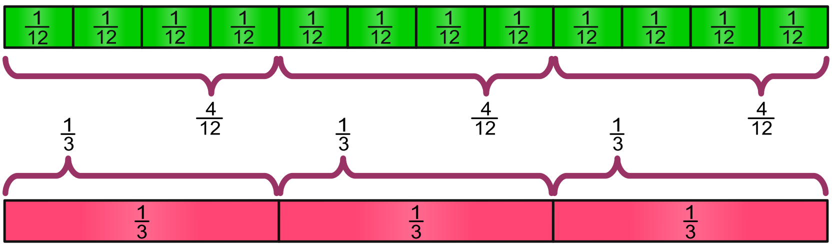
If a fraction has the denominator 12 and a numerator that is divisible by 6, we can simplify the fraction by 6. For example:

$$\frac{6}{12} = \frac{6 \div 6}{12 \div 6} = \frac{1}{2} \quad \text{and} \quad \frac{18}{12} = \frac{18 \div 6}{12 \div 6} = \frac{3}{2}.$$



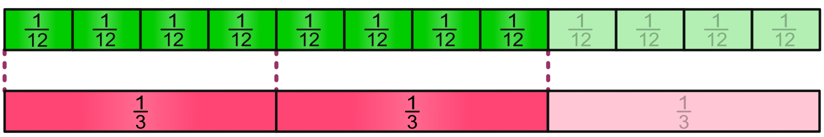


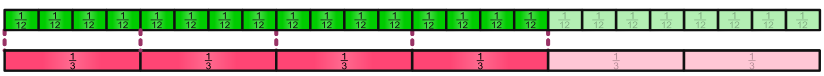
12 is also divisible by 4.
Therefore, we can divide the 12 twelfths into groups of four.
On one whole, we then have 3 equal parts.
Each of these parts is thus equal to $\mathbf{\frac{1}{3}}$. So: $\mathbf{\frac{4}{12}} = \mathbf{\frac{1}{3}}$.



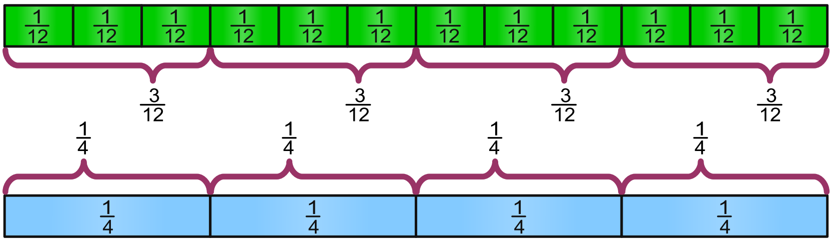
If a fraction has the denominator 12 and a numerator that is divisible by 4, we can simplify the fraction by 4. For example:

$$\frac{8}{12} = \frac{8 \div 4}{12 \div 4} = \frac{2}{3} \quad \text{and} \quad \frac{16}{12} = \frac{16 \div 4}{12 \div 4} = \frac{4}{3}.$$





12 is also divisible by 3. Therefore, we can combine 3 twelfths into *one* part and obtain fourths.



If a fraction has the denominator 12 and a numerator that is divisible by 3, we can simplify the fraction by 3. For example:

$$\frac{6}{12} = \frac{6 \div 3}{12 \div 3} = \frac{2}{4} \quad \text{and} \quad \frac{9}{12} = \frac{9 \div 3}{12 \div 3} = \frac{3}{4}.$$

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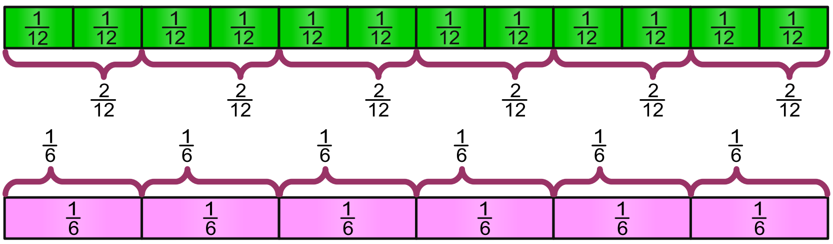
12 is also divisible by 2. Therefore, we can combine 2 twelfths into *one* part and obtain sixths.



If a fraction has the denominator 12 and a numerator that is divisible by 2, we can simplify the fraction by 2. For example:

$$\frac{6}{12} = \frac{6 \div 2}{12 \div 2} = \frac{3}{6} \quad \text{and} \quad \frac{28}{12} = \frac{28 \div 2}{12 \div 2} = \frac{14}{6}.$$

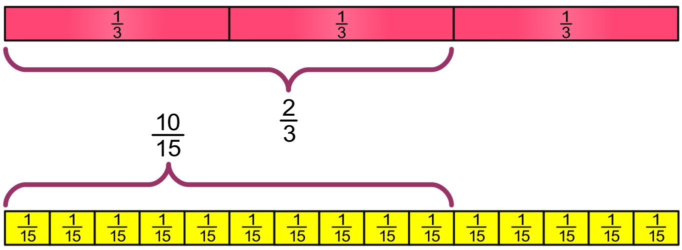
1.4 Simplifying as the inverse of expanding

We can also think of simplifying as the inverse of expanding. Expanding means, for example: We divide each third of $\frac{2}{3}$ into 5 equal parts. Each of these parts then has one fifth the size of a third. Therefore, we need 5 times as many parts for a fraction of equal size.

So: $\frac{2}{3} = \frac{2 \times 5}{3 \times 5} = \frac{10}{15}$.

Now we can go the other way and group every 5 of the 10 fifteenths into larger parts. These parts are then 5 times as large as the previous ones, and therefore we only need $10 \div 5 = 2$ parts for a fraction of equal size.

So: $\frac{10}{15} = \frac{10 \div 5}{15 \div 5} = \frac{2}{3}$.



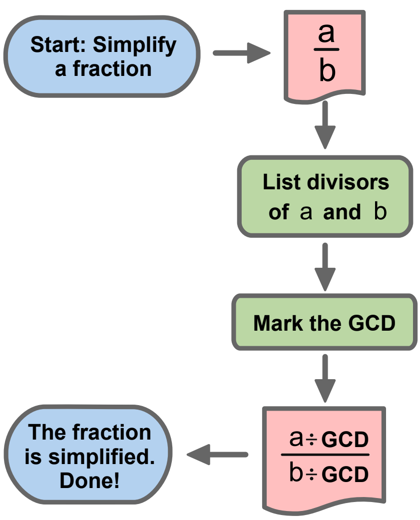
1.5 Simplifying a Fraction

Calculating with simplified fractions is usually easier than calculating with unsimplified fractions. We usually simplify a fraction using the greatest common divisor of the numerator and the denominator.
If we are given a fraction, we can proceed as follows:

1. Write down all divisors of the numerator.
2. Write down all divisors of the denominator.
3. Mark the greatest common divisor GCD of numerator and denominator.
4. Divide numerator and denominator by the greatest common divisor GCD.

From now on, whenever we talk about simplifying a fraction, we always mean simplifying a fraction using the GCD.

We can also represent this procedure as a flowchart.



Let's look at some examples:

1.6 Examples

Example 1

The fraction $\mathbf{\frac{8}{12}}$ has numerator 8 and denominator 12.

The divisors of 8 are: $1; 2; 4; 8$.

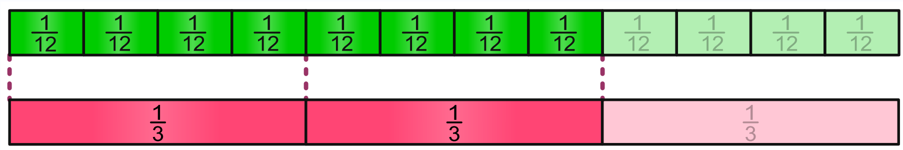
The divisors of 12 are: $1; 2; 3; 4; 6; 12$.

We have marked the GCD in red.

Now we divide the numerator and denominator by the GCD.

$$\frac{8}{12} = \frac{8 \div 4}{12 \div 4} = \frac{2}{3}$$

So if we simplify $\mathbf{\frac{8}{12}}$, we get $\mathbf{\frac{2}{3}}$.



Example 2

The fraction $\mathbf{\frac{15}{18}}$ has numerator 15 and denominator 18.

The divisors of 15 are: $1; 3; 5; 15$.

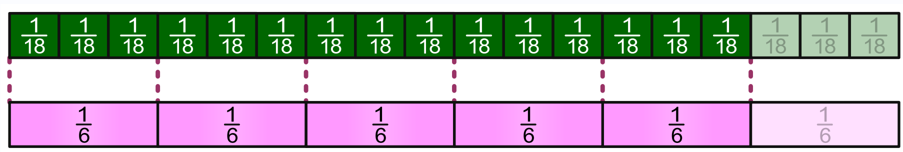
The divisors of 18 are: $1; 2; 3; 6; 9; 18$.

We have marked the GCD in red.

Now we divide the numerator and denominator by the GCD.

$$\frac{15}{18} = \frac{15 \div 3}{18 \div 3} = \frac{5}{6}$$

So if we simplify $\mathbf{\frac{15}{18}}$, we get $\mathbf{\frac{5}{6}}$.



Example 3

The fraction $\frac{108}{756}$ has numerator 108 and denominator 756.

The divisors of 108 are: $1; 2; 3; 4; 6; 9; 12; 18; 27; 36; 54; 108$.

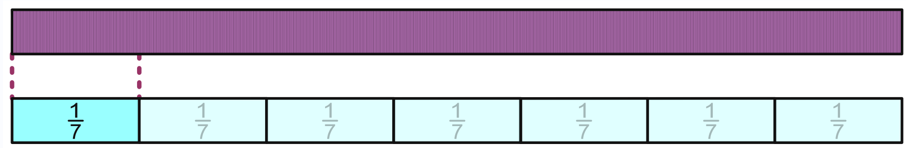
The divisors of 756 are: $1; 2; 3; 4; 6; 7; 9; 12; 14; 18; 21; 27; 28; 36; 42; 54; 63; 84; 108; 126; 189; 252; 378; 756$.

We have marked the GCD in red.

Now we divide numerator and denominator by the GCD.

$$\frac{108}{756} = \frac{108 : 108}{756 : 108} = \frac{1}{7}$$

So, if we reduce the fraction $\frac{108}{756}$, we get $\frac{1}{7}$.



Example 4

The fraction $\frac{40}{81}$ has numerator 40 and denominator 81.

The divisors of 40 are: $1; 2; 4; 5; 10; 20; 40$.

The divisors of 81 are: $1; 3; 9; 27; 81$.

We have marked the GCD in red.

Since the greatest common divisor of numerator and denominator is 1, this fraction cannot be reduced in a meaningful way.